35^{th} BALKAN MATHEMATICAL OLYMPIAD
Belgrade, Serbia (May 9, 2018)

Problem 1.

A quadrilateral $ABCD$ is inscribed in a circle k, where $AB > CD$ and AB is not parallel to CD. Point M is the intersection of the diagonals AC and BD and the perpendicular from M to AB intersects the segment AB at the point E. If EM bisects the angle CED, prove that AB is a diameter of the circle k.

Problem 2.

Let q be a positive rational number. Two ants are initially at the same point X in the plane. In the n-th minute $(n = 1, 2, \dots)$ each of them chooses whether to walk due north, east, south or west and then walks the distance of q^n metres. After a whole number of minutes, they are at the same point in the plane (not necessarily X), but have not taken exactly the same route within that time. Determine all possible values of q.

Problem 3.

Alice and Bob play the following game: They start with two non-empty piles of coins. Taking turns, with Alice playing first, each player chooses a pile with an even number of coins and moves half of the coins of this pile to the other pile. The game ends if a player cannot move, in which case the other player wins.

Determine all pairs (a, b) of positive integers such that if initially the two piles have a and b coins respectively, then Bob has a winning strategy.

Problem 4.

Find all primes p and q such that $3p^{q-1} + 1$ divides $11^p + 17^p$.

Time allowed: 4 hours and 30 minutes.
Each problem is worth 10 points.

S O L U T I O N S

Problem 1.

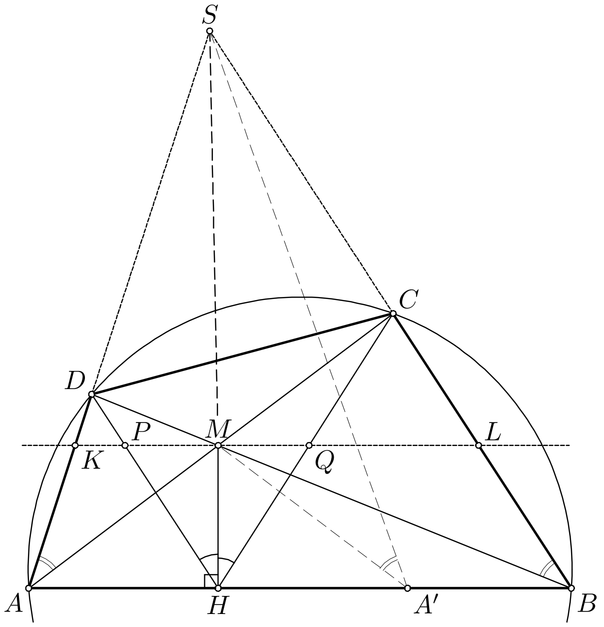
Let the line through M parallel to AB meet the segments AD, DH, BC, CH at points K, P, L, Q, respectively. Triangle HPQ is isosceles, so $MP = MQ$. Now from

$$\frac{MP}{BH} = \frac{DM}{DB} = \frac{KM}{AB} \quad \text{and} \quad \frac{MQ}{AH} = \frac{CM}{CA} = \frac{ML}{AB}$$

we obtain $AH/HB = KM/ML$.

Let the lines AD and BC meet at point S and let the line SM meet AB at H'. Then $AH'/H'B = KM/ML = AH/HB$, so $H' \equiv H$, i.e. S lies on the line MH.

The quadrilateral $ABCD$ is not a trapezoid, so $AH \neq BH$. Consider the point A' on the ray HB such that $HA' = HA$. Since $\sphericalangle SA'M = \sphericalangle SAM = \sphericalangle SBM$, quadrilateral $A'BSM$ is cyclic and therefore $\sphericalangle ABC = \sphericalangle A'BS = \sphericalangle A'MH = \sphericalangle AMH = 90° - \sphericalangle BAC$, which implies that $\sphericalangle ACB = 90°$.



Problem 2.

Answer: $q = 1$.

Let $x_A^{(n)}$ (resp. $x_B^{(n)}$) be the x-coordinates of the first (resp. second) ant's position after n minutes. Then $x_A^{(n)} - x_A^{(n-1)} \in \{q^n, -q^n, 0\}$, and so $x_A^{(n)}, x_B^{(n)}$ are given by polynomials in q with coefficients in $\{-1, 0, 1\}$. So if the ants meet after n minutes, then

$$0 = x_A^{(n)} - x_B^{(n)} = P(q),$$

where P is a polynomial with degree at most n and coefficients in $\{-2, -, 1, 0, 1, 2\}$. Thus if $q = \frac{a}{b}$ $(a, b \in \mathbb{N})$, we have $a \mid 2$ and $b \mid 2$, i.e. $q \in \{\frac{1}{2}, 1, 2\}$.

It is clearly possible when $q = 1$.

We argue that $q = \frac{1}{2}$ is not possible. Assume that the ants diverge for the first time after the kth minute, for $k \geqslant 0$. Then

$$\left| x_B^{(k+1)} - x_A^{(k+1)} \right| + \left| y_B^{(k+1)} - y_A^{(k+1)} \right| = 2q^k. \tag{1}$$

But also $\left| x_A^{(\ell+1)} - x_A^{(\ell)} \right| + \left| y_A^{(\ell+1)} - y_A^{(\ell)} \right| = q^\ell$ for each $l \geqslant k+1$, and so

$$\left| x_A^{(n)} - x_A^{(k+1)} \right| + \left| y_A^{(n)} - y_A^{(k+1)} \right| \leqslant q^{k+1} + q^{k+2} + \ldots + q^{n-1}. \tag{2}$$

and similarly for the second ant. Combining (1) and (2) with the triangle inequality, we obtain for any $n \geqslant k+1$

$$\left| x_B^{(n)} - x_A^{(n)} \right| + \left| y_B^{(n)} - y_A^{(n)} \right| \geqslant 2q^k - 2\left(q^{k+1} + q^{k+2} + \ldots + q^{n-1} \right),$$

which is strictly positive for $q = \frac{1}{2}$. So for any $n \geqslant k+1$, the ants cannot meet after n minutes. Thus $q \neq \frac{1}{2}$.

Finally, we show that $q = 2$ is also not possible. Suppose to the contrary that there is a pair of routes for $q = 2$, meeting after n minutes. Now consider rescaling the plane by a factor 2^{-n}, and looking at the routes in the opposite direction. This would then be an example for $q = 1/2$ and we have just shown that this is not possible.

Solution 2.

Consider the ants' positions α_k and β_k after k steps in the complex plane, assuming that their initial positions are at the origin and that all steps are parallel to one of the axes. We have $\alpha_{k+1} - \alpha_k = a_k q^k$ and $\beta_{k+1} - \beta_k = b_k q^k$ with $a_k, b_k \in \{1, -1, i, -i\}$.

If $\alpha_n = \beta_n$ for some $n > 0$, then

$$\sum_{k=0}^{n-1} (a_k - b_k) q^k = 0, \quad \text{where} \quad a_k - b_k \in \{0, \pm 1 \pm i, \pm 2, \pm 2i\}.$$

Note that the coefficient $a_k - b_k$ is always divisible by $1+i$ in Gaussian integers: indeed,

$$c_k = \frac{a_k - b_k}{1+i} \in \{0, \pm 1, \pm i, \pm 1 \pm i\}.$$

Canceling $1+i$, we obtain $c_0 + c_1 q + \cdots + c_{n-1} q^{n-1} = 0$. Therefore if $q = \frac{a}{b}$ $(a, b \in \mathbb{N})$, we have $a \mid c_0$ and $b \mid c_{n-1}$ in Gaussian integers, which is only possible if $a = b = 1$.

Problem 3.

By $v_2(n)$ we denote the largest nonnegative integer r such that $2^r \mid n$.

A position (a, b) (i.e. two piles of sizes a and b) is said to be *k-happy* if $v_2(a) = v_2(b) = k$ for some integer $k \geqslant 0$, and k-unhappy if $\min\{v_2(a), v_2(b)\} = k < \max\{v_2(a), v_2(b)\}$. We shall prove that Bob has a winning strategy if and only if the initial position is k-happy for some even k.

- Given a 0-happy position, the player in turn is unable to play and loses.
- Given a k-happy position (a, b) with $k \geqslant 1$, the player in turn will transform it into one of the positions $(a + \frac{1}{2}b, \frac{1}{2}b)$ and $(b + \frac{1}{2}a, \frac{1}{2}a)$, both of which are $(k-1)$-happy because $v_2(a + \frac{1}{2}b) = v_2(\frac{1}{2}b) = v_2(b + \frac{1}{2}a) = v_2(\frac{1}{2}a) = k - 1$.

Therefore, if the starting position is k-happy, after k moves they will get stuck at a 0-happy position, so Bob will win if and only if k is even.

- Given a k-unhappy position (a, b) with k odd and $v_2(a) = k < v_2(b) = \ell$, Alice can move to position $(\frac{1}{2}a, b + \frac{1}{2}a)$. Since $v_2(\frac{1}{2}a) = v_2(b + \frac{1}{2}a) = k - 1$, this position is $(k-1)$-happy with $2 \mid k - 1$, so Alice will win.

- Given a k-unhappy position (a, b) with k even and $v_2(a) = k < v_2(b) = \ell$, Alice must not play to position $(\frac{1}{2}a, b + \frac{1}{2}a)$, because the new position is $(k-1)$-happy and will lead to Bob's victory. Thus she must play to position $(a + \frac{1}{2}b, \frac{1}{2}b)$. We claim that this position is also k-unhappy. Indeed, if $\ell > k + 1$, then $v_2(a + \frac{1}{2}b) = k < v_2(\frac{1}{2}b) = \ell - 1$, whereas if $\ell = k + 1$, then $v_2(a + \frac{1}{2}b) > v_2(\frac{1}{2}b) = k$.

Therefore a k-unhappy position is winning for Alice if k is odd, and drawing if k is even.

Problem 4.

Answer: $(p, q) = (3, 3)$.

For $p = 2$ it is directly checked that there are no solutions. Assume that $p > 2$.

Observe that $N = 11^p + 17^p \equiv 4 \pmod 8$, so $8 \nmid 3p^{q-1} + 1 > 4$. Consider an odd prime divisor r of $3p^{q-1} + 1$. Obviously, $r \notin \{3, 11, 17\}$. There exists b such that $17b \equiv 1 \pmod r$. Then $r \mid b^p N \equiv a^p + 1 \pmod r$, where $a = 11b$. Thus $r \mid a^{2p} - 1$, but $r \nmid a^p - 1$, which means that $\mathrm{ord}_r(a) \mid 2p$ and $\mathrm{ord}_r(a) \nmid p$, i.e. $\mathrm{ord}_r(a) \in \{2, 2p\}$.

Note that if $\mathrm{ord}_r(a) = 2$, then $r \mid a^2 - 1 \equiv (11^2 - 17^2)b^2 \pmod r$, which gives $r = 7$ as the only possibility. On the other hand, $\mathrm{ord}_r(a) = 2p$ implies $2p \mid r - 1$. Thus, all prime divisors of $3p^{q-1} + 1$ other than 2 or 7 are congruent to 1 modulo $2p$, i.e.

$$3p^{q-1} + 1 = 2^\alpha 7^\beta p_1^{\gamma_1} \cdots p_k^{\gamma_k}, \qquad (*)$$

where $p_i \notin \{2, 7\}$ are prime divisors with $p_i \equiv 1 \pmod{2p}$.

We already know that $\alpha \leqslant 2$. Also, note that

$$\frac{11^p + 17^p}{28} = 11^{p-1} - 11^{p-2}17 + 11^{p-3}17^2 - \cdots + 17^{p-1} \equiv p \cdot 4^{p-1} \pmod 7,$$

so $11^p + 17^p$ is not divisible by 7^2 and hence $\beta \leqslant 1$.

If $q = 2$, then $(*)$ becomes $3p+1 = 2^\alpha 7^\beta p_1^{\gamma_1} \cdots p_k^{\gamma_k}$, but $p_i \geqslant 2p+1$, which is only possible if $\gamma_i = 0$ for all i, i.e. $3p + 1 = 2^\alpha 7^\beta \in \{2, 4, 14, 28\}$, which gives us no solutions.

Thus $q > 2$, which implies $4 \mid 3p^{q-1} + 1$, i.e. $\alpha = 2$. Now the right hand side of $(*)$ is congruent to 4 or 28 modulo p, which gives us $p = 3$. Consequently $3^q + 1 \mid 6244$, which is only possible for $q = 3$. The pair $(p, q) = (3, 3)$ is indeed a solution.